number results which enumerates the combinations with which the problem is concerned. The problems to which the method is applicable are those which are, directly or indirectly, associated with lattices, and it is remarkable that, for the most part, they are such as have been hitherto regarded as unassailable by the processes of pure mathematics.

If a problem be presented for solution, it may be a difficult matter to design the function and the operation, the combination of which furnishes the solution. The plan here adopted is to consider certain functions and operations, and then to inquire into the nature of the associated problems. The author has attempted to place the method on a sure foundation, to give illustrative examples of gradually increasing complexity, and to indicate some promising lines of future investigation. The only published work connected with the subject is the author's paper, entitled "A New Method in Combinatory Analysis, with application to Latin Squares and Associated Questions," which will be found in the 'Trans. Camb. Phil. Soc.,' vol. xvi, Part IV, p. 262.

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"Über Reihen auf der Convergenzgrenze." Von EMANUEL LASKER, Dr. Philos. Communicated by Major MACMAHON, F.R.S. Received March 15,—Read April 5, 1900.

(Abstract.)

The essay is divided into three chapters.

A limit operation of any kind, for instance, simple or multiple integration; the formation of infinite sums or products; or any combination of these operations gives rise to certain typical considerations which form the subject of the first chapter. To fix the ideas, let

$$F(x) = u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots,$$

where the $u_i$ are analytical functions of $x$; and let $x = \xi$ be a point on the curve which forms the boundary of the region of convergence of the series. Let C be a curve within the region of convergence terminating in $x = \xi$. In that case $F(\xi)$ is generally different from

$$\lim. F(x),$$

$$(\lim. x = \xi),$$

$x$ varying on the curve C. It is a matter of the greatest importance to examine the relations between these two mathematical conceptions. If $u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots$ converges where $x = \xi$, $F(\xi)$ has a definite

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meaning, but it does not thence follow that $\underset{(\lim.\ x\ =\ \xi)}{\lim.\ \mathrm{F}(x)}$ has also a definite meaning.

But whenever the sum $U_1+U_2+\ldots\ldots+U_n+\ldots\ldots$ of the maxima values, which $u_1, u_2, \ldots\ldots u_n, \ldots\ldots$ assume upon C, is convergent, the expression

$$\underset{(\lim.\ x\ =\ \xi)}{\mathrm{F}(x)}$$

has always a definite limit which coincides with $\mathrm{F}(\xi)$. It is very easy to derive from this theorem a multitude of others. For instance, the maximum value of $x^n(1-x)^\lambda$ being

$$\frac{n^n\lambda^\lambda}{(n+\lambda)^{n+\lambda}},$$

$\lambda$ being positive and $x$ varying within the circle whose centre is 0, and whose radius is unity

$$(1-x)^\lambda \underset{n\ =\ 0,\ \ldots\ \infty}{\Sigma c_n x^n} \ =\ \underset{n\ =\ 0,\ \ldots\ \infty}{\Sigma c_n x^n (1-x)^\lambda}$$

has, for lim. $x = 1$, zero as limit whenever $\Sigma\, c_n/n^\lambda$ converges.

If $u_1+u_2+\ldots\ldots+u_n+\ldots\ldots$ be divergent when $x = \xi$, $\underset{(\lim.\ x\ =\ \xi)}{\mathrm{F}(x)}$ will have $\infty$ as limit, if a certain condition be satisfied; viz., the curves described by $u_n$, $x$ varying upon C, should lie in an angle $\omega$, less than $\pi$, whose vertex is 0. This criterion is principally important on account of its connection with a certain theorem, the "type theorem."

Two sequences

$$u_1, u_2, \ldots\ldots u_n, \ldots\ldots$$

$$v_1, v_2, \ldots\ldots v_n, \ldots\ldots$$

are said to be of the same "type" if $u_n/v_n$, lim. $n = \infty$, lim. $x = \xi$, tends uniformly towards a finite limit $\rho$ different from zero. If then $u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots$ be divergent when $x = \xi$ and the $u_n$ satisfy the above condition

$$\frac{u_1+u_2+\ \ldots\ldots\ +u_n}{v_1+v_2+\ \ldots\ldots\ +v_n} \quad (\lim.\ x\ =\ \xi)$$

will tend towards $\rho$.

The second chapter deals with an application of the type theorem. A power series $\underset{n\ =\ 0,\ \ldots\ \infty}{\Sigma c_n x^n}$ whose coefficients are such that

$$\frac{c_0+c_1+\ \ldots\ldots\ +c_n}{n^\lambda}$$

has for indefinitely increasing values of $n$, as limit $\rho$, tends multiplied by $(1 - x)^{\lambda}$, towards the limit $\rho\,\Gamma(\lambda + 1)$. This proposition, which follows immediately from the type theorem, gives for $\lambda = 0$ Abel's theorem; for $\lambda = 1$, the theorem of Frobenius ('Crelle,' 1878); and leads to consequences in regard to the integrals of linear differential equations with rational coefficients. The principal result of this chapter, is that the singularities of a function, defined by a power series $\Sigma c_n x^n$, can be found and their nature analysed by the examination only, for values of $n$ tending towards $\infty$, of the sequence $c_n$ and of other sequences derived from this one.

A new conception is introduced in the third chapter, that of an "asymptotical region." An asymptotical region encloses always a point $x = a$, of essential singularity, of a function $F(x)$ and consists of an ordinary region enclosing $a$ with an infinity of co-holes in it, not enclosing $a$, but approaching it indefinitely. The object of the author is to throw some light on the manner in which a function behaves in the vicinity of a point of essential singularity. It is shown that if

$$F(x) = \Sigma \frac{b_n}{(x - c_n)} m_n,$$

$a$ being $\infty$, and the $m_n$ denoting finite integers, an asymptotical region can generally be constructed in which lim. $F(x) = 0$ if lim. $x = \infty$; and that this proposition is convertible. The new conception is applied to the theory of transcendental integral functions as founded by Weierstrass, Laguerre, Poincaré, &c. If $G(x)$ denotes a function of class zero, a certain asymptotical region will belong to $G'/G$ in the above sense. If $H(x) = G(x) + c_1 G'(x) + c_2 G''(x) + \ldots\ldots$ where the $c_i$ are any constants, such that the power series $\Sigma c_i t^i$ $(i = 1, \ldots\ldots \infty)$ has a non-vanishing radius of convergence, $H(x)$ will again be a transcendental integral function of class zero; and the asymptotical region belonging to $H(x)$ will be the same as that belonging to $G(x)$.

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"Further Note on the Influence of the Temperature of Liquid Air on Bacteria." By ALLAN MACFADYEN, M.D., and S. ROWLAND, M.A. Communicated by LORD LISTER, P.R.S. Received April 3,—Read April 5, 1900.

In a previous communication* it was shown that no appreciable influence was exerted upon the vital properties of bacteria when exposed for 20 hours to the temperature of liquid air (−183° C. to

* 'Roy. Soc. Proc.,' February 1, 1900.